p-CAPACITY VS SURFACE-AREA

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ABSTRACT. This paper is devoted to exploring the relationship between the $[1,n)\ni p$-capacity and the surface-area in $\mathbb{R}^{n\geq 2}$ which especially shows: if $\Omega\subset\mathbb{R}^n$ is a convex, compact, smooth set with its interior $\Omega^\circ\neq\emptyset$ and the mean curvature $H(\partial\Omega,\cdot)>0$ of its boundary $\partial\Omega$ then

$$\left(\frac{n(p-1)}{p(n-1)}\right)^{p-1}\leq\frac{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)}{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}}\leq\left(\sqrt[n-1]{\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}}\right)^{p-1}\quad\forall\quad p\in(1,n)$$

whose limits $1\leftarrow p$ & $p\to n$ imply

$$1=\frac{cap_1(\Omega)}{\mathrm{area}(\partial\Omega)}\ \&\ \int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\geq 1,$$

thereby not only discovering that the new best known constant is roughly half as far from the one conjectured by Pólya-Szegö in [25, (2)] but also extending the Pólya-Szegö inequality in [25, (5)], with both the conjecture and the inequality being stated for the electrostatic capacity of a convex solid in $\mathbb{R}^3$.

1. OVERVIEW

Given a compact set Ω in the $2\leq n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with the standard volume and surface-area elements dv and $d\sigma$. The variational $[1,n)\ni p$-capacity of Ω is defined by

$$\mathrm{cap}_p(\Omega)=\inf\left\{\int_{\mathbb{R}^n}|\nabla f|^p\,dv:\quad f\in C_c^\infty(\mathbb{R}^n)\ \&\ f(x)\geq 1\ \forall\ x\in\Omega\right\},$$

where $C_c^\infty(\mathbb{R}^n)$ is the class of all infinitely differentiable functions with compact support in $\mathbb{R}^n$. Equivalently, the above infimum can be taken over either all $f\in C_c^\infty(\mathbb{R}^n)$ with $f=1$ in a neighbourhood of Ω, or all Lipschitz functions u on $\mathbb{R}^n$ with $f=1$ in a neighbourhood of Ω (cf. [11, pp. 27-28]).

As a set function on compact subsets of $\mathbb{R}^n$, $\mathrm{cap}_p(\cdot)$ enjoys the following basic properties (a) through (f) (cf. [11, pp. 28-32] and [20, Lemma 2.2.5]):

(a) Boundarization – if Ω is a compact subset of $\mathbb{R}^n$ with non-empty boundary $\partial\Omega$ then

$$\mathrm{cap}_p(\partial\Omega)=\mathrm{cap}_p(\Omega).$$

(b) Monotonicity – if Ω_1 and Ω_2 are compact subsets of $\mathbb{R}^n$ with $\Omega_1\subseteq\Omega_2$ then

$$\mathrm{cap}_p(\Omega_1)\leq\mathrm{cap}_p(\Omega_2).$$

(c) Continuity – if $(\Omega_j)_{j=1}^\infty$ is a decreasing sequence of compact subsets of $\mathbb{R}^n$ then

$$\mathrm{cap}_p(\cap_{j=1}^\infty\Omega_j)=\lim_{j\to\infty}\mathrm{cap}_p(\Omega_j).$$

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(d) Ball capacity – if $B(x,r)=\{y\in\mathbb{R}^n:\ |y-x|\le r\}$ and σ_{n-1} is the surface area of the origin-centred unit ball $B(0,1)$ then

$$\mathrm{cap}_p(B(x,r))=r^{n-p}\Big(\frac{p-1}{n-p}\Big)^{1-p}\sigma_{n-1}.$$

(e) Geometric endpoint – if Ω is a compact subset of $\mathbb{R}^n$ and area$(\cdot)$ stands for the surface-area of a set in $\mathbb{R}^n$ then

$$\mathrm{cap}_1(\Omega)=\inf\{\mathrm{area}(\partial\Lambda):\ \Omega\subset\Lambda\cup\partial\Lambda \text{ with bound open } \Lambda \text{ and smooth } \partial\Lambda\}.$$

(f) Physical interpretation – if Ω is a compact subset of $\mathbb{R}^{n\ge 3}$, then $\mathrm{cap}_2(\Omega)$ is the maximal charge which can be placed on Ω when the electrical potential of the vector field created by this charge is controlled by 1, namely,

$$\mathrm{cap}_2(\Omega)=\sup\Big\{\mu(\Omega):\ \text{measure}\,\mu\,\text{with}\,\mathrm{supp}(\mu)\subseteq\Omega\ \&\ \int_{\mathbb{R}^n}|x-y|^{2-n}\,\frac{d\mu(y)}{(n-2)\sigma_{n-1}}\le 1\ \forall\,x\in\mathbb{R}^n\backslash\Omega\Big\}.$$

Motivated by Pólya's 1947 paper [25] as well as (a)&(e) above, this article stems from discovering the relationship between the p-capacity and the surface-area (via the mean curvature). The details for such a discovery are provided in §2&§3 whose summary is shown in the sequel:

(h) Surface area to variational capacity (§2) – In Theorem 2.1 we use the convexity of level set of $(1,n)\ni p$-equilibrium potential and a minimizing technique to gain (2.4), a sharp convexity type inequality, linking the normalized variational capacity, the normalized surface area and the normalized volume and consequently deriving that $(\frac{n(p-1)}{p(n-1)})^{p-1}$ times $(\frac{n-p}{n-1})$-th power of the normalized surface area is the asymptotically sharp lower bound of the normalized variational capacity, whence having half-solved [1] the Pólya-Szegö conjecture (for $\mathrm{cap}_2(\cdot)$ in $\mathbb{R}^3$) that *of all convex bodies, with a given surface area, the circular disk has the minimum capacity.*;

(i) Variational capacity to surface area (§3) – In Theorem 3.1 we employ a level set formulation of the inverse mean curvature flow (generated by a kind of 1-equilibrium potential) to achieve (3.3), a log-convexity type inequality involving the normalized variational capacity, the normalized surface area and the normalized Willmore functional for the mean curvature and consequently revealing that the product of both $(\frac{p-1}{n-1})$-th power of the normalized Willmore functional for the mean curvature and $(\frac{n-p}{n-1})$-th power of the normalized surface area is the optimal upper bound of the normalized variational capacity, thereby extending the Pólya-Szegö principle (for $\mathrm{cap}_2(\cdot)$ in $\mathbb{R}^3$) that *unless the convex solid is a ball the capacity is less than the mean-curvature-radius*.

Naturally, a combination of (2.5) in Theorem 2.1 and (3.4) in Theorem 3.1 derives that if $\Omega\subset\mathbb{R}^n$ is a convex, compact, smooth set with its interior $\Omega^c\neq\emptyset$ and the mean curvature $H(\partial\Omega,\cdot)>0$ of its boundary $\partial\Omega$ then

(j)

$$\Big(\frac{n(p-1)}{p(n-1)}\Big)^{p-1}\le\frac{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)}{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}}\le\left(\sqrt[n-1]{\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}}\right)^{p-1}\quad\forall\quad p\in(1,n)$$

whose limiting cases $1\leftarrow p$ & $p\to n$ surprisingly yield the extremal case of (e) (cf. [19]) and the Willmore inequality (cf. [2, 29, 1]) as seen below:

[1]Namely, the new best known constant is roughly half as far from the conjectured one.

(k)
$$1=\frac{cap_1(\Omega)}{\text{area}(\partial\Omega)}\ \ \&\ \int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\geq 1.$$

2. Surface-area to p-capacity

In [27, p.12] (cf. [25]) Pólya-Szegö conjectured that for any convex compact subset Ω of $\mathbb{R}^3$ one has

$$\text{cap}_2(\Omega)\geq\left(4\sqrt{\frac{2}{\pi}}\right)\sqrt{\text{area}(\partial\Omega)} \tag{2.1}$$

with equality if and only if Ω is a two-dimensional disk in $\mathbb{R}^3$. Here it is perhaps worth pointing out that if $\Omega\subset\mathbb{R}^2$ then area$(\partial\Omega)$ is replaced by two times of the two-dimensional Lebesgue measure of Ω.

The first remarkable result approaching the conjecture was obtained in Pólya-Szegö's 1951 monograph: [27, p.165,(4)] (as a sequel to the work presented in their 1945 paper [26]) via suitable symmetrization and projection for any given convex compact set $\Omega\subset\mathbb{R}^3$:

$$\text{cap}_2(\Omega)\geq\left(\frac{4}{\sqrt{\pi}}\right)\sqrt{\text{area}(\partial\Omega)}. \tag{2.2}$$

Since then, no improvement has been made on (2.2) and of course (2.1) has not yet been verified - see [16, 4, 5, 14] for an up-to-date report on this research. In the sequel, with the help of the isocapacitary inequality for the volume vol$(\cdot)$ of a level set of the equilibrium potential of an arbitrary convex compact set $\Omega\subset\mathbb{R}^3$ we show

$$\text{cap}_2(\Omega)\geq\left(\frac{3\sqrt{\pi}}{2}\right)\sqrt{\text{area}(\partial\Omega)}, \tag{2.3}$$

whence finding that (2.3) holds the nearly middle place between (2.1) and (2.2) in the sense of

$$\begin{cases}4\sqrt{\frac{2}{\pi}}>\frac{3\sqrt{\pi}}{2}>\frac{4}{\sqrt{\pi}};\\ 4\sqrt{\frac{2}{\pi}}-\frac{3\sqrt{\pi}}{2}=0.532857...;\\ \frac{3\sqrt{\pi}}{2}-\frac{4}{\sqrt{\pi}}=0.401922....\end{cases}$$

As a matter of fact, we discover the brand-new sharp convexity type inequality (2.4) (for the surface-area, the variational capacity and the volume) whose by-product (2.5) is much more general than (2.3).

Theorem 2.1. *Let Ω be a convex compact subset of $\mathbb{R}^n$ with area$(\partial\Omega)>0$. Then*

$$\frac{n(p-1)}{p(n-1)}\left(\frac{\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}}{\left(\frac{cap_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{p-1}}+\frac{n-p}{p(n-1)}\left(\frac{\left(\frac{vol(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}}{\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}}\right)^{n}\leq 1\ \ \forall\ \ p\in(1,n) \tag{2.4}$$

holds with equality if and only if Ω is a ball. Consequently

$$\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}\leq\left(\frac{cap_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)\left(\frac{p(n-1)}{n(p-1)}\right)^{p-1}\ \ \forall\ p\in(1,n), \tag{2.5}$$

which is asymptotically optimal in the sense that if $p \to 1$ or $p \to n$ in (2.5) *then*

$$\textit{area}(\partial\Omega) = \textit{cap}_1(\Omega) \quad \textit{or} \quad 1 = 1. \tag{2.6}$$

Proof. First of all, since $\mathrm{area}(\partial\Omega) > 0$ and Ω is convex, it follows from [19] that $\mathrm{cap}_1(\Omega) = \mathrm{area}(\partial\Omega) > 0$. In accordance with [32, Theorem 3.2], if $1 \le p_1 < p_2 < n$ then there is a constant $c(p_1, p_2, n) > 0$ depending only on (p_1, p_2, n) such that

$$(\mathrm{cap}_{p_1}(\Omega))^{\frac{1}{n-p_1}} \le c(p_1, p_2, n)(\mathrm{cap}_{p_2}(\Omega))^{\frac{1}{n-p_2}}.$$

Upon choosing $p_1 = 1 < p_2 = p < n$, one gets $\mathrm{cap}_p(\Omega) > 0$.

Next, we verify (2.4) through considering two situations.

Situation 1: suppose that the interior Ω° of Ω is not empty and the boundary $\partial\Omega$ of Ω is of C^1-smoothness. In accordance with [3, 17], there is a unique $(1, n) \ni p$-equilibrium potential u of Ω (not only smooth in $\Omega^c = \mathbb{R}^n \setminus \Omega$ but also continuous in $\mathbb{R}^n \setminus \Omega^\circ$) such that:

- $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω^c;
- $u|_{\partial\Omega} = 1$;
- $\lim_{|x|\to\infty} u(x) = 0$;
- $0 < u < 1$ in Ω^c;
- $|\nabla u| \ne 0$ in Ω^c;
- $$\mathrm{cap}_p(\Omega) = \int_{\mathbb{R}^n\setminus\Omega} |\nabla u|^p\, dv = \int_{\{x\in\mathbb{R}^n:\ u(x)=t\}} |\nabla u|^{p-1}\, d\sigma \quad \forall \quad t \in (0, 1);$$
- if u is set to be 1 on Ω then $\{x \in \mathbb{R}^n :\ u(x) \ge t\}$ is convex and $\{x \in \mathbb{R}^n :\ u(x) = t\}$ is smooth for any $t \in (0, 1)$.

Consequently, we can utilize the well-known monotonicity for the area function of convex domains, the Hölder inequality and the co-area formula to get

$$\begin{aligned}
&\mathrm{area}(\partial\Omega)\\
&\le \mathrm{area}(\{x \in \mathbb{R}^n :\ u(x) = t\})\\
&= \int_{\{x\in\mathbb{R}^n:\ u(x)=t\}} d\sigma\\
&\le \left(\int_{\{x\in\mathbb{R}^n:\ u(x)=t\}} |\nabla u|^{p-1}\, d\sigma\right)^{\frac{1}{p}} \left(\int_{\{x\in\mathbb{R}^n:\ u(x)=t\}} |\nabla u|^{-1}\, d\sigma\right)^{\frac{p-1}{p}}\\
&= (\mathrm{cap}_p(\Omega))^{\frac{1}{p}} \left(-\frac{d}{dt}\mathrm{vol}(\{x \in \mathbb{R}^n :\ u(x) \ge t\})\right)^{\frac{p-1}{p}},
\end{aligned}$$

and accordingly,

$$\left(\frac{\mathrm{area}(\partial\Omega)}{(\mathrm{cap}_p(\Omega))^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} \le -\frac{d}{dt}\mathrm{vol}(\{x \in \mathbb{R}^n :\ u(x) \ge t\}), \tag{2.7}$$

where

$$\mathrm{vol}(\{x \in \mathbb{R}^n :\ u(x) \ge t\})$$

is the Lebesgue measure of the upper level set $\{x \in \mathbb{R}^n :\ u(x) \ge t\}$. Recalling the Poincaré-Mazya isocapacitary inequality (cf. [27] for $p = 2$ and [20] for $p \in (1, n)$)

$$\frac{\mathrm{vol}(\{x \in \mathbb{R}^n :\ u(x) \ge t\})}{n^{-1}\sigma_{n-1}} \le \left(\frac{\mathrm{cap}_p(\{x \in \mathbb{R}^n :\ u(x) \ge t\})}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}}$$

and using (a) - the boundarization of $\mathrm{cap}_p(\cdot)$ to achieve the following formula (cf. [27, 24] for $p = 2$)

$$\begin{aligned}
&\mathrm{cap}_p(\{x \in \mathbb{R}^n : \ u(x) \ge t\})\\
&= \mathrm{cap}_p(\{x \in \mathbb{R}^n : \ u(x) = t\})\\
&= \int_{\{x\in\mathbb{R}^n:\ u(x)=t\}} \left(t^{-1}|\nabla u|\right)^{p-1} d\sigma\\
&= t^{1-p}\mathrm{cap}_p(\Omega),
\end{aligned}$$

we obtain via integrating both sides of (2.7) over the interval $(t, 1)$

$$\begin{aligned}
&(1-t)\left(\frac{\mathrm{area}(\partial\Omega)}{\left(\mathrm{cap}_p(\Omega)\right)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}\\
&\le \mathrm{vol}(\{x \in \mathbb{R}^n : \ u(x) \ge t\}) - \mathrm{vol}(\Omega)\\
&\le \left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{\mathrm{cap}_p(\{x \in \mathbb{R}^n : \ u(x) \ge t\})}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}} - \mathrm{vol}(\Omega)\\
&= \left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{t^{1-p}\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}} - \mathrm{vol}(\Omega).
\end{aligned}$$

Note that the above estimate is valid for any $t \in [0, 1]$. But if

$$t \in \left[1, \left(\frac{\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}}{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{1-p}}\right]$$

then

$$(1-t)\left(\frac{\mathrm{area}(\partial\Omega)}{\left(\mathrm{cap}_p(\Omega)\right)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} \le 0 \le \left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{t^{1-p}\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}} - \mathrm{vol}(\Omega)$$

and hence one has:

$$(1-t)\left(\frac{\mathrm{area}(\partial\Omega)}{\left(\mathrm{cap}_p(\Omega)\right)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} \le \left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{t^{1-p}\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}} - \mathrm{vol}(\Omega) \quad \forall \quad t \in \left[0, \left(\frac{\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}}{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{1-p}}\right].$$

Suppose t_0 is the critical point of the following function

$$t \mapsto \phi(t) = (1-t)\left(\frac{\mathrm{area}(\partial\Omega)}{\left(\mathrm{cap}_p(\Omega)\right)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} - \left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{t^{1-p}\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}} + \mathrm{vol}(\Omega).$$

Then solving $\phi'(t_0)=0$ and using the classical isoperimetric inequality one gets

$$t_0=\left(\frac{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}}{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{1-p}}\le\left(\frac{\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}}{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{1-p}},$$

whence deriving

$$(1-t_0)\left(\frac{\mathrm{area}(\partial\Omega)}{\left(\mathrm{cap}_p(\Omega)\right)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}\le\left(\frac{\sigma_{n-1}}{n}\right)\left(\frac{t_0^{1-p}\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{n}{n-p}}-\mathrm{vol}(\Omega),$$

which implies

$$\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\le\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n}{n-1}}-\left(\frac{1-t_0}{n^{-1}\sigma_{n-1}}\right)\left(\frac{\mathrm{area}(\partial\Omega)}{(\mathrm{cap}_p(\Omega))^{\frac{1}{p}}}\right)^{\frac{p}{p-1}},$$

namely,

$$1-t_0\le\left(\frac{n-p}{n(p-1)}\right)t_0\left(1-\frac{\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)}{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n}{n-1}}}\right),$$

and then (2.4) via a further computation with t_0.

Situation 2: suppose that Ω is a general convex compact subset of $\mathbb{R}^n$. For this setting there is a sequence of convex compact sets $(\Omega_j)_{j=1}^\infty$ such that $\Omega_j^\circ\neq\emptyset$, $\partial\Omega_j$ is of C^1-smoothness, and Ω_j decreases to Ω. Since (2.4) and (2.5) are valid for Ω_j, an application of the continuity for area$(\cdot)$, vol$(\cdot)$, and cap$_p(\cdot)$ acting on convex compact sets ensures that (2.4) is true for such Ω.

After that, we check the equality case of (2.4). If Ω is a ball, then an application of both (d) and the identity

$$\frac{n(p-1)}{p(n-1)}+\frac{n-p}{p(n-1)}=1$$

makes equality of (2.4) happen. Conversely, if equality of (2.4) occurs for all $p\in(1,n)$, then

$$\frac{n(p-1)}{p(n-1)}\left(\frac{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}}{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)^{\frac{1}{n-p}}}\right)^{\frac{n-p}{p-1}}+\frac{n-p}{p(n-1)}\left(\frac{\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}}{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}}\right)^{n}=1\ \ \forall\ \ p\in(1,n).$$

Upon letting $p\to1$ in this last equality and using the known fact that (cf. [22, 19])

$$\liminf_{p\to1}\mathrm{cap}_p(\Omega)=\mathrm{cap}_1(\Omega)=\mathrm{area}(\partial\Omega)$$

we obtain

$$\left(\frac{\mathrm{vol}(\Omega)}{n^{-1}\sigma_{n-1}}\right)^{\frac{1}{n}}=\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}},$$

namely, equality of the isoperimetric inequality holds for Ω, thereby finding that Ω is a ball.

Finally, let us deal with (2.5) and its limiting cases. Note that the second term of the left-hand-side of (2.4) is non-negative. So, (2.5) follows immediately from (2.4). Moreover, the first identity of (2.6), as the limit case $p\to1$ of (2.5), is well-known; see also [19], [8] and

[20, Lemma 2.2.5]. To see the second identity of (2.6), let $B(0, R_0)$ be an origin-symmetric ball containing Ω. Using (2.5) and (b)&(d) we find

$$1 = \liminf_{p\to n}\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}} \le \liminf_{p\to n}\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)\left(\frac{p(n-1)}{n(p-1)}\right)^{p-1} \le \liminf_{p\to n} R_0^{n-p} = 1,$$

as desired. □

Remark 2.2. *Below are two comments on (2.5) of independent interest:*

(i) *In accordance with* [15, Proposition 1.1], *if Ω is a convex compact subset of $\mathbb{R}^{n\ge 3}$ with $\Omega^\circ \neq \emptyset$ and smooth $\partial\Omega$, and u is the $p = 2$-equllibrium potential of Ω, then an application of the fact that*

$$x \mapsto v(x) = \int_{\partial\Omega} |x-y|^{2-n}\,\frac{d\sigma(y)}{(n-2)\sigma_{n-1}}$$

is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ (cf. [21]*) gives*

$$v(x) = v_\infty\big((n-2)\sigma_{n-1})\big)^{-1}|x|^{2-n} + O(|x|^{1-n}) \quad as \quad |x| \to \infty,$$

where

$$v_\infty = \int_{\partial\Omega} v|\nabla u|\,d\sigma.$$

Note that (cf. [21]*)*

$$v(x) = \big((n-2)\sigma_{n-1})\big)^{-1} area(\partial\Omega)|x|^{2-n} + O(|x|^{1-n}) \quad as \quad |x| \to \infty.$$

So, one has

$$\begin{aligned} &\big((n-2)\sigma_{n-1})\big)^{-1} area(\partial\Omega)\\ &= v_\infty\\ &= \int_{\partial\Omega} v|\nabla u|\,d\sigma \\ &\le \big(\max_{x\in\partial\Omega} v(x)\big)\int_{\partial\Omega}|\nabla u|\,d\sigma\\ &= \big(\max_{x\in\partial\Omega} v(x)\big) cap_2(\Omega). \end{aligned} \tag{2.8}$$

Using the well-known layer-cake formula under $d\sigma$, one finds

$$\begin{aligned} &v(x)\Big((n-2)\sigma_{n-1})\Big)\\ &= \int_0^\infty \sigma(\{y\in\partial\Omega:\ |x-y|^{2-n}\ge t\})\,dt\\ &= \left(\int_0^r + \int_r^\infty\right)\sigma(\{y\in\partial\Omega:\ |x-y|^{2-n}\ge t\})\,dt\\ &\le area(\partial\Omega) r + (n-2)\sigma_{n-1} r^{\frac{1}{2-n}}. \end{aligned}$$

Minimizing the last quantity, one gets that

$$r = \left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{2-n}{n-1}}$$

derives

$$\int_{\partial\Omega}|x-y|^{2-n}\,\frac{d\sigma(y)}{\sigma_{n-1}} \le (n-1)\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}}. \tag{2.9}$$

This (2.9), *along with* (2.8), *yields*

$$\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-2}{n-1}} \le (n-1)\left(\frac{cap_2(\Omega)}{(n-2)\sigma_{n-1}}\right). \tag{2.10}$$

The inequality (2.10) is weaker than the case $p = 2$ of (2.5). However, (2.10) can be strengthened upon demonstrating the following conjecture

$$\int_{\partial\Omega} |x-y|^{2-n}\,\frac{d\sigma(y)}{\sigma_{n-1}} \le \left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}} \quad \forall \quad x \in \partial\Omega, \tag{2.11}$$

with equality if and only if Ω is a ball; see [18, p.249,(4)], [21] *and* [7] *for some information related to* (2.11).

(ii) *The higher dimensional extension of the variational principle presented in* [28, Theorem 1.1] *derives that if Ω is a convex compact subset of $\mathbb{R}^n$ with $\Omega^\circ \ne \emptyset$ and smooth $\partial\Omega$ then*

$$\frac{(n-2)\sigma_{n-1}}{cap_2(\Omega)} \le \frac{\int_{\partial\Omega}\int_{\partial\Omega} |x-y|^{2-n}\,d\sigma(x)d\sigma(y)}{\big(area(\partial\Omega)\big)^2}. \tag{2.12}$$

A combination of (2.9) and (2.12) gives (2.10).

3. p-capacity to surface-area

From [25, (5)] it follows that if $n = 3$ and Ω is a convex compact subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and its mean curvature $H(\partial\Omega,\cdot) > 0$ then one has the following Pólya-Szegö inequality for the electrostatic capacity and the mean radius:

$$\frac{\mathrm{cap}_2(\Omega)}{4\pi} \le \int_{\partial\Omega} H(\partial\Omega,\cdot)\,\frac{d\sigma(\cdot)}{4\pi} \tag{3.1}$$

with equality if Ω is a ball. This result has been extended by Freire-Schwartz to any outer-minimizing $\partial\Omega$ in $\Omega^c = \mathbb{R}^{n\ge3} \setminus \Omega$, i.e., $\Omega \subseteq \Lambda \Rightarrow \mathrm{area}(\partial\Omega) \le \mathrm{area}(\partial\Lambda)$ (cf. [6, Theorem 2]):

$$\frac{\mathrm{cap}_2(\Omega)}{(n-2)\sigma_{n-1}} \le \int_{\partial\Omega} H(\partial\Omega,\cdot)\,\frac{d\sigma(\cdot)}{\sigma_{n-1}} \tag{3.2}$$

with equality if and only if Ω is a ball. As a higher dimensional star-shaped generalization of (3.1), we have the following result whose (3.3) under $p = 2$ is a nice parallelism of (3.2) since the outer-minimizing and the star-shaped are not mutually inclusive; see also [10], and whose (3.4) discovers an optimal relation between the variational capacity and the surface area via the Willmore functional of the mean curvature (cf. [1, Corollary 2] for $(p,n) = (2,3)$).

Theorem 3.1. *Let Ω be a smooth, star-shaped, compact subset of $\mathbb{R}^n$ with $\Omega^\circ \ne \emptyset$ and $H(\partial\Omega,\cdot) > 0$. Then*

$$\frac{cap_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}} \le \begin{cases} \int_{\partial\Omega}\Big(H(\partial\Omega,\cdot)\Big)^{p-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}} & as \;\; 2 \le p < n; \\ \left(\int_{\partial\Omega}\Big(H(\partial\Omega,\cdot)\Big)^{q-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{p-1}{q-1}}\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{q-p}{q-1}} & as \;\; 1 < p \le 2 \le q < n, \end{cases} \tag{3.3}$$

where the first inequality becomes an equality if and only if Ω is a ball. Consequently

$$\frac{cap_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}} \le \left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}\left(\int_{\partial\Omega}\Big(H(\partial\Omega,\cdot)\Big)^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{p-1}{n-1}} \quad \forall \quad p \in (1,n) \tag{3.4}$$

holds with equality if and only if Ω is a ball. Moreover, the limit settings $p \to 1$ or $p \to n$ in (3.4) *produce*

$$cap_1(\Omega) \le area(\partial\Omega) \quad or \quad 1 \le \int_{\partial\Omega} (H(\partial\Omega, \cdot))^{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}}. \tag{3.5}$$

Proof. First of all, recall that a classic solution of inverse mean curvature flow in $\mathbb{R}^n$ is a smooth collection $F : M^{n-1} \times [0, T) \mapsto \mathbb{R}^n$ of closed hypersurfaces evolving by

$$\frac{\partial}{\partial t} F(x,t) = \frac{\tau(x,t)}{H(x,t)} \quad \forall \quad (x,t) \in M^{n-1} \times [0,T), \tag{3.6}$$

where

$$H(x,t) = \operatorname{div}(\tau(x,t)) > 0 \quad \text{and} \quad \tau(x,t)$$

are the mean curvature and the outward unit normal vector of the embedded hypersurface $M_t = F(M^{n-1}, t)$. According to Gerhardt [9] (or Urbas [30, 31]), one has that for any smooth, closed, star-sharped, initial hypersurface of positive mean curvature, equation (3.6) has a unique smooth solution for all times and the rescaled hypersurfaces M_t converge exponentially to a unique sphere as $t \to \infty$.

According to Moser's description (cf. [23]) of the inverse mean curvature flow (whose weak formulation was studied in Huisken-Ilimanen's papers [12, 13]), we see that a level set formulation of the above parabolic evolution problem for hypersufaces in $\mathbb{R}^n$ with the initial hypersurface $M_0 = \Sigma = \partial\Omega$ produces a non-negative smooth function u in Ω^c such that:

- $\operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = |\nabla u|$ in Ω^c;
- $u|_{\partial\Omega} = 0$;
- $u = t$ on $M_t = \Sigma_t$;
- $|\nabla u| \neq 0$ in Ω^c;
- $H(\Sigma_t, \cdot) = (n-1)^{-1} |\nabla u(\cdot)|$ on Σ_t;
- $\operatorname{area}(\Sigma_t) = e^t \operatorname{area}(\partial\Omega)\ \forall\ t \ge 0$.

This function u may be treated as a kind of 1-equilibrium potential of Ω - more precisely - if $u_p = \exp\big(\frac{u}{1-p}\big)$ obeys $\operatorname{div}(|\nabla u_p|^{p-2} \nabla u_p) = 0$ in Ω^c and $u_p|_{\partial\Omega} = 1$ then $(1-p)\log u_p \to u$ locally uniformally in Ω^c as $p \to 1$; see [23, Theorem 1.1].

According to (a) and the determination of pcap$(\cdot)$ in terms of the $(1,n) \ni p$-equilibrium potential of Ω, we have

$$\operatorname{cap}_p(\Omega) = \operatorname{cap}_p(\partial\Omega) \le \inf_f \int_{\mathbb{R}^n \setminus \Omega^\circ} |\nabla f|^p \, dv \tag{3.7}$$

where the infimum is taken over all functions $f = \psi \circ g$ that have the above-described level hypersurfaces $(\Sigma_t)_{t \ge 0}$ and enjoy the property that ψ is a one-variable function with $\psi(0) = 0$ and $\psi(\infty) = 1$ and g is a non-negative function on $\mathbb{R}^n \setminus \Omega^\circ$ with $g|_{\partial\Omega} = 0$ and $\lim_{|x| \to \infty} g(x) = \infty$. Note that the co-area formula yields

$$\int_{\mathbb{R}^n \setminus \Omega^\circ} |\nabla f|^p \, dv = \int_0^\infty |\psi'(t)|^p \left(\int_{\Sigma_t} |\nabla g|^{p-1} \, d\sigma_t \right) dt.$$

In the above and below, $d\sigma_t$ is the surface-area-element on Σ_t. So, upon choosing

$$\begin{cases} g = u; \\ U_p(t) = \int_{\Sigma_t} |\nabla u|^{p-1} \frac{d\sigma_t}{\sigma_{n-1}}; \\ \psi(t) = V_p(t) = \dfrac{\int_0^t \big(U_p(s)\big)^{\frac{1}{1-p}} \, ds}{\int_0^\infty \big(U_p(s)\big)^{\frac{1}{1-p}} \, ds}, \end{cases}$$

we utilize (3.7) to achieve

$$\frac{\mathrm{cap}_p(\Omega)}{\sigma_{n-1}} \le \int_0^\infty U_p(t)\Big|\frac{d}{dt}V_p(t)\Big|^p\,dt,$$

whence finding

$$\frac{\mathrm{cap}_p(\Omega)}{\sigma_{n-1}} \le \left(\int_0^\infty (U_p(t))^{\frac{1}{1-p}}\,dt\right)^{1-p}. \tag{3.8}$$

Next, let us work out the growth of $U_p(\cdot)$.

Case 1: $p \in [2, n)$. Under this assumption, utilizing [13, Lemma 1.2, (ii)&(v)], an integration-by-part, the inequality

$$(H(\Sigma_t,\cdot))^2 - (n-1)|\mathrm{II}_t|^2 \le 0$$

with

$$0 < H(\Sigma_t,\cdot) = (n-1)^{-1}|\nabla u|$$

and II_t being the mean curvature and the second fundamental form on Σ_t respectively, the differentiation under the integral, we obtain

$$\begin{aligned}
&\frac{d}{dt}U_p(t)\\
&= \frac{d}{dt}\left(\frac{(n-1)^{p-1}}{\sigma_{n-1}}\int_{\Sigma_t}(H(\Sigma_t,\cdot))^{p-1}\,d\sigma_t\right)\\
&= \frac{(n-1)^{p-1}}{\sigma_{n-1}}\int_{\Sigma_t}\left((p-1)(H(\Sigma_t,\cdot))^{p-2}\Big(\frac{d}{dt}H(\Sigma_t,\cdot)\Big) + (H(\Sigma_t,\cdot))^{p-1}\right)d\sigma_t\\
&= \frac{(n-1)^{p-1}}{\sigma_{n-1}}\int_{\Sigma_t}\left(1-(p-1)\Big(\frac{|\mathrm{II}_t|}{H(\Sigma_t,\cdot)}\Big)^2 - (p-2)\Big|\nabla(H(\Sigma_t,\cdot))^{-1}\Big|^2\right)(H(\Sigma_t,\cdot))^{p-1}\,d\sigma_t\\
&\le \frac{n-p}{(n-1)\sigma_{n-1}}\int_{\Sigma_t}|\nabla u|^{p-1}\,d\sigma_t\\
&= \Big(\frac{n-p}{n-1}\Big)U_p(t),
\end{aligned}$$

whence discovering the following inequality through an integration

$$U_p(t) \le U_p(0)\exp\Big(t\big(\frac{n-p}{n-1}\big)\Big). \tag{3.9}$$

Using (3.8)-(3.9) we get

$$\frac{\mathrm{cap}_p(\Omega)}{\sigma_{n-1}} \le U_p(0)\left(\frac{(n-1)(p-1)}{n-p}\right)^{1-p}$$

whence reaching the inequality in (3.3) under $2 \le p < n$.

Case 2: $1 < p \le 2 \le q < n$. Under this situation, we use the Hölder inequality to achieve

$$\int_{\Sigma_t}|\nabla u|^{p-1}\frac{d\sigma_t}{\sigma_{n-1}} \le \left(\int_{\Sigma_t}|\nabla u|^{q-1}\frac{d\sigma_t}{\sigma_{n-1}}\right)^{\frac{p-1}{q-1}}\left(\frac{\mathrm{area}(\Sigma_t)}{\sigma_{n-1}}\right)^{\frac{q-p}{q-1}}.$$

Now, employing the estimate for $q \in [2, n)$ and the definition of U_p, we obtain

$$U_p(t) \le \left(\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{q-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{p-1}{q-1}}\left(\frac{\mathrm{area}(\Sigma_t)}{\sigma_{n-1}}\right)^{\frac{q-p}{q-1}}\exp\Big(t\big(\frac{n-p}{n-1}\big)\Big).$$

Bringing this last inequality into (3.8), along with

$$\mathrm{area}(\Sigma_t) = e^t\,\mathrm{area}(\partial\Omega),$$

we arrive at the second inequality of (3.3).

Case 3: equality of (3.3). If Ω is a ball, then a direct computation gives equality of (3.3). Conversely, if the inequality $\le$ in (3.3) becomes an equality, then the above-established differential inequalities for U_p force

$$(H(\Sigma_t,\cdot))^2-(n-1)|\mathrm{II}_t|^2=0\quad\text{on}\quad\Sigma_t,$$

which in turn ensures that Σ_t consists of the union of disjoint spheres. Since Σ_t is generated by a smooth solution of the inverse mean curvature flow in $\mathbb{R}^n$, Σ_t must be a single sphere. Consequently, Ω is a ball.

After that, (3.4) and its equality case follow from (3.3) and its equality case as well as the following estimate (based on the Hölder inequality)

$$\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{q-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\le\left(\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{q-1}{n-1}}\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-q}{n-1}}\quad\forall\quad q\in(1,n).$$

Finally, let us check (3.5). On the one hand, letting $p\to1$ in (3.4) yields the Mazya inequality (cf. [20, p.149, Lemma 2.2.5]):

$$cap_1(\Omega)\le\mathrm{area}(\partial\Omega).$$

On the other hand, choosing $0<r<R$ with $B(x_0,r)\subseteq\Omega\subseteq B(x_0,R)$, we utilize the properties (b)&(d) of $\mathrm{cap}_p(\cdot)$ to derive

$$r^{n-p}\le\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\le R^{n-p},$$

whence achieving

$$\lim_{p\to n}\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}=1.$$

This, together with letting $p\to n$ in (3.4), derives the Willmore inequality (cf. [29, p. 87] or [2] for immersed hypersurfaces in $\mathbb{R}^n$):

$$1\le\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{n-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}.$$

□

Remark 3.2. *Two comments are in order:*

(i) *Let Ω be a smooth compact subset of $\mathbb{R}^n$ with $\Omega^\circ\neq\emptyset$ and $H(\partial\Omega,\cdot)>0$. If $\partial\Omega$ is outer-minimizing, then one has the $(1,n)\ni p$-Aleksandrov-Fenchel inequality:*

$$\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}\le\begin{cases}\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{p-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}} & as\ \ 2\le p<n;\\ \left(\int_{\partial\Omega}\left(H(\partial\Omega,\cdot)\right)^{q-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{p-1}{q-1}}\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{q-p}{q-1}} & as\ \ 1<p\le2\le q<n,\end{cases}\tag{3.10}$$

where the first inequality becomes an equality if and only if Ω is a ball.

In fact, using the known 2-Aleksandrov-Fenchel inequality (cf. [6, Theorem 2(b)]*)*

$$\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-2}{n-1}}\le\int_{\partial\Omega}H(\partial\Omega,\cdot)\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\tag{3.11}$$

and the Hölder inequality, we gain

$$\int_{\partial\Omega}H(\partial\Omega,\cdot)\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\le\left(\int_{\partial\Omega}(H(\partial\Omega,\cdot))^{p-1}\,\frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{1}{p-1}}\left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{p-2}{p-1}}\quad\forall\quad p\in[2,n),$$

whence implying (3.10). *If the first inequality of* (3.10) *becomes equality, then equality of* (3.11) *is valid, and hence* Ω *is a ball. Of course, the converse follows from a direct computation.*

(ii) *An application of* (3.2), (3.10) *and the Hölder inequality derives that if* $\Omega \subset \mathbb{R}^{n\geq 3}$ *is a smooth compact set with* $\Omega^\circ \neq \emptyset$ *and* $\partial\Omega$ *being outer-minimizing as well as having* $H(\partial\Omega, \cdot) > 0$ *then one has the following log-convexity type inequality for the electrostatic capacity, the surface area and the Willmore functional:*

$$\frac{cap_2(\Omega)}{(n-2)\sigma_{n-1}} \leq \left(\frac{area(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-2}{n-1}} \left(\int_{\partial\Omega} (H(\partial\Omega, \cdot))^{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}} \tag{3.12}$$

with equality if and only if Ω *is a ball. Interestingly and naturally,* (3.12) *and* (3.4) *under* $p = 2$ *complement each other thanks to the relative independence between the outer-minimizing and the star-shaped.*

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